

## A NOTE ON LATTICES IN SEMI-STABLE REPRESENTATIONS

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ABSTRACT. Let  $p \geq 3$  be a prime,  $K$  a finite extension over  $\mathbb{Q}_p$  and  $G := \text{Gal}(\bar{K}/K)$ . We extend Kisin's theory on  $\varphi$ -modules of finite  $E(u)$ -height to give a new classification of  $G$ -stable  $\mathbb{Z}_p$ -lattices in semi-stable representations.

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## 1. INTRODUCTION

This note serves as a new idea to classify lattices in the semi-stable representations. Let  $k$  be a perfect field of characteristic  $p > 2$ ,  $W(k)$  its ring of Witt vectors,  $K_0 = W(k)[\frac{1}{p}]$ ,  $K/K_0$  a finite totally ramified extension and  $G := \text{Gal}(\bar{K}/K)$ . For many technical reasons, we are interested in classifying  $G$ -stable  $\mathbb{Z}_p$ -lattices in semi-stable  $p$ -adic Galois representations, via linear algebra data like admissible filtered  $(\varphi, N)$ -modules in Fontaine's theory. Many important steps have been made in this direction. For example, Fontaine and Laffaille' theory [FL82] on strongly divisible  $W(k)$ -lattices in filtered  $(\varphi, N)$ -modules, Breuil's theory on strongly divisible  $S$ -lattices ([Bre02], [Liu07a]), and Berger and Breuil's theory on Wach modules ([BB07]). Unfortunately, these classifications always have some restrictions (on the absolute ramification index, Hodge-Tate weights, etc). Based on Kisin's theory in [Kis06], the aim of this paper is to provide a classification without these restrictions (at least for  $p > 2$ ).

More precisely, let  $E(u)$  be an Eisenstein polynomial for a fixed uniformizer  $\pi$  of  $K$ ,  $K_\infty = \cup_{n \geq 1} K(\sqrt[n]{\pi})$ ,  $G_\infty = \text{Gal}(\bar{K}/K_\infty)$  and  $\mathfrak{S} = W(k)[[u]]$ . We equip  $\mathfrak{S}$  with the endomorphism  $\varphi$  which acts via Frobenius on  $W(k)$ , and sends  $u$  to

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$u^p$ . Let  $\text{Mod}_{/\mathfrak{S}}^{r, \text{fr}}$  denote the category of finite free  $\mathfrak{S}$ -modules  $\mathfrak{M}$  equipped with a  $\varphi$ -semi-linear map  $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$  such that the cokernel of  $\mathfrak{S}$ -linear map  $1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{M}$  is killed by  $E(u)^r$ . Objects in  $\text{Mod}_{/\mathfrak{S}}^{r, \text{fr}}$  are called  $\varphi$ -modules of  $E(u)$ -height  $r$  or *Kisin modules*. In [Kis06], Kisin proved that any  $G_{\infty}$ -stable  $\mathbb{Z}_p$ -lattice  $T$  in a semi-stable Galois representation comes from a Kisin module (See Theorem 2.1.1 for details). Obviously, extra data have to be added if one would like to extend the classification of  $G_{\infty}$ -stable lattices to the classification of  $G$ -stable lattices. Our ideal is to imitate the theory of  $(\varphi, \Gamma)$ -modules. But  $G_{\infty}$  is not a normal subgroup of  $G$  and there is no natural  $G$ -action on  $\mathfrak{S}$ . To remedy this, we construct a  $\mathfrak{S}$ -algebra  $\widehat{\mathcal{R}}$  inside  $W(R)$  such that  $\widehat{\mathcal{R}}$  is stable under Frobenius and the  $G$ -action. Furthermore, the  $G$ -action on  $\widehat{\mathcal{R}}$  factors through  $\hat{G} := \text{Gal}(K_{\infty, p^{\infty}}/K)$  where  $K_{\infty, p^{\infty}}$  is the Galois closure of  $K_{\infty}$ . The construction of  $\widehat{\mathcal{R}}$  allows us to define  $(\varphi, \hat{G})$ -module which is Kisin module  $(\mathfrak{M}, \varphi)$  with extra semi-linear  $\hat{G}$ -action on  $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$  compatible with Frobenius (see Definition 2.2.3 for details). Our main result in this note is that the category of  $G$ -stable  $\mathbb{Z}_p$ -lattices in semi-stable representations with Hodge-Tate weights in  $\{0, \dots, r\}$  is anti-equivalent to the category of  $(\varphi, \hat{G})$ -modules of  $E(u)$ -height  $r$ .

Just as any integral version of  $p$ -adic Hodge theory before,  $(\varphi, \hat{G})$ -modules will help us better to understand the reduction of semi-stable representations, and this will be discussed in forthcoming work. On the other hand, so far we do not fully understand the structure of  $\widehat{\mathcal{R}}$ . In fact,  $\widehat{\mathcal{R}}$  seems quite complicated (See Example 3.2.3). So at least at this stage, it seems that our theory only serves as a theoretic approach. We hope we can simplify this theory in the future by further exploring the structure of  $\widehat{\mathcal{R}}$ , such that we could provide more explicit examples or carry out some concrete computations by  $(\varphi, \hat{G})$ -modules.

## 2. PRELIMINARY AND THE MAIN RESULT

**2.1. Kisin Modules.** Recall that  $k$  is a perfect field of characteristic  $p > 2$ ,  $W(k)$  its ring of Witt vectors,  $K_0 = W(k)[\frac{1}{p}]$ ,  $K/K_0$  a finite totally ramified extension and  $e = e(K/K_0)$  the absolute ramification index. Throughout this paper we fix a uniformiser  $\pi \in K$  with Eisenstein polynomial  $E(u)$ . Recall that  $\mathfrak{S} = W(k)[[u]]$  is equipped with a Frobenius endomorphism  $\varphi$  via  $u \mapsto u^p$  and the natural Frobenius on  $W(k)$ . Throughout this paper we reserve  $\varphi$  to denote various Frobenius structures. A  $\varphi$ -module (over  $\mathfrak{S}$ ) is an  $\mathfrak{S}$ -module  $\mathfrak{M}$  equipped with a  $\varphi$ -semi-linear map  $\varphi : \mathfrak{M} \rightarrow \mathfrak{M}$ . A morphism between two objects  $(\mathfrak{M}_1, \varphi_1)$ ,  $(\mathfrak{M}_2, \varphi_2)$  is a  $\mathfrak{S}$ -linear morphism compatible with the  $\varphi_i$ . Denote by  $\text{Mod}_{/\mathfrak{S}}^{r, \text{fr}}$  the category of  $\varphi$ -modules of  $E(u)$ -height  $r$  in the sense that  $\mathfrak{M}$  is finite free <sup>1</sup> over  $\mathfrak{S}$  and the cokernel of  $\varphi^*$  is killed by  $E(u)^r$ , where  $\varphi^*$  is the  $\mathfrak{S}$ -linear map  $1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{M}$ . Object in  $\text{Mod}_{/\mathfrak{S}}^{r, \text{fr}}$  is also called *Kisin module (of height <sup>2</sup>  $r$ )*.

Let  $R = \varprojlim \mathcal{O}_{\bar{K}}/p$  where the transition maps are given by Frobenius. By the universal property of the Witt vectors  $W(R)$  of  $R$ , there is a unique surjective projection map  $\theta : W(R) \rightarrow \widehat{\mathcal{O}}_{\bar{K}}$  to the  $p$ -adic completion of  $\mathcal{O}_{\bar{K}}$ , which lifts the

<sup>1</sup>This is a somewhat ad hoc definition because we only concern finite free  $\mathfrak{S}$ -modules here. In fact, one may only require that  $\mathfrak{M}$  is of  $\mathfrak{S}$ -finite type when define  $\varphi$ -modules of finite  $E(u)$ -height, especially, when study  $p$ -power torsion representations.

<sup>2</sup>Throughout this paper, the height is always  $E(u)$ -height. So we always omit “ $E(u)$ ”.

the projection  $R \rightarrow \mathcal{O}_{\bar{K}}/p$  onto the first factor in the inverse limit. Let  $\pi_n \in \bar{K}$  be a  $p^n$ -th root of  $\pi$ , such that  $(\pi_{n+1})^p = \pi_n$ ; write  $\underline{\pi} = (\pi_n)_{n \geq 0} \in R$  and let  $[\underline{\pi}] \in W(R)$  be the Teichmüller representative. We embed the  $W(k)$ -algebra  $W(k)[u]$  into  $W(R)$  by the map  $u \mapsto [\underline{\pi}]$ . This embedding extends to an embedding  $\mathfrak{S} \hookrightarrow W(R)$ , and, as  $\theta([\underline{\pi}]) = \pi$ ,  $\theta|_{\mathfrak{S}}$  is the map  $\mathfrak{S} \rightarrow \mathcal{O}_K$  sending  $u$  to  $\pi$ . This embedding is compatible with Frobenius endomorphisms.

Denote by  $\mathcal{O}_{\mathcal{E}}$  the  $p$ -adic completion of  $\mathfrak{S}[\frac{1}{u}]$ . Then  $\mathcal{O}_{\mathcal{E}}$  is a discrete valuation ring with residue field the Laurent series ring  $k((u))$ . We write  $\mathcal{E}$  for the field of fractions of  $\mathcal{O}_{\mathcal{E}}$ . If  $\text{Fr}R$  denotes the field of fractions of  $R$ , then the inclusion  $\mathfrak{S} \hookrightarrow W(R)$  extends to an inclusion  $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\text{Fr}R)$ . Let  $\mathcal{E}^{\text{ur}} \subset W(\text{Fr}R)[\frac{1}{p}]$  denote the maximal unramified extension of  $\mathcal{E}$  contained in  $W(\text{Fr}R)[\frac{1}{p}]$ , and  $\mathcal{O}^{\text{ur}}$  its ring of integers. Since  $\text{Fr}R$  is easily seen to be algebraically closed, the residue field  $\mathcal{O}^{\text{ur}}/p\mathcal{O}^{\text{ur}}$  is the separable closure of  $k((u))$ . We denote by  $\widehat{\mathcal{E}^{\text{ur}}}$  the  $p$ -adic completion of  $\mathcal{E}^{\text{ur}}$ , and by  $\widehat{\mathcal{O}^{\text{ur}}}$  its ring of integers.  $\widehat{\mathcal{E}^{\text{ur}}}$  is also equal to the closure of  $\mathcal{E}^{\text{ur}}$  in  $W(\text{Fr}R)[\frac{1}{p}]$ . We write  $\mathfrak{S}^{\text{ur}} = \widehat{\mathcal{O}^{\text{ur}}} \cap W(R) \subset W(\text{Fr}R)$ . We regard all these rings as subrings of  $W(\text{Fr}R)[\frac{1}{p}]$ .

Recall that  $K_{\infty} = \bigcup_{n \geq 0} K(\pi_n)$ , and  $G_{\infty} = \text{Gal}(\bar{K}/K_{\infty})$ .  $G_{\infty}$  acts continuously on  $\mathfrak{S}^{\text{ur}}$  and  $\mathcal{E}^{\text{ur}}$  and fixes the subring  $\mathfrak{S} \subset W(R)$ . Finally, we denote by  $\text{Rep}_{\mathbb{Z}_p}(G_{\infty})$  the category of continuous  $\mathbb{Z}_p$ -linear representations of  $G_{\infty}$  on finite free  $\mathbb{Z}_p$ -modules.

For any Kisin module  $(\mathfrak{M}, \varphi)$ , one can associate a  $\mathbb{Z}_p[G_{\infty}]$ -module:

$$T_{\mathfrak{S}}(\mathfrak{M}) := \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}^{\text{ur}}).$$

One can show that  $T_{\mathfrak{S}}(\mathfrak{M})$  is finite free over  $\mathbb{Z}_p$  and  $\text{rank}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M})) = \text{rank}_{\mathfrak{S}}(\mathfrak{M})$  (see for example, Corollary (2.1.4) in [Kis06]). Let  $V$  be a continuous linear representation of  $G := \text{Gal}(\bar{K}/K)$  on a finite dimensional  $\mathbb{Q}_p$ -vector space.  $V$  is called of  $E(u)$ -height  $r$  if there exists a  $G_{\infty}$ -stable  $\mathbb{Z}_p$ -lattices  $T \subset V$  and a Kisin module  $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}}^{r, \text{fr}}$  such that  $T \simeq T_{\mathfrak{S}}(\mathfrak{M})$ . We refer [Fon94b] to the notion of *semi-stable*  $p$ -adic representations<sup>3</sup>. The following theorem summarizes the known results on the relation between semi-stable representations and representations of finite  $E(u)$ -height.

**Theorem 2.1.1** ([Kis06]). (1) *The functor  $T_{\mathfrak{S}} : \text{Mod}_{\mathfrak{S}}^{r, \text{fr}} \rightarrow \text{Rep}_{\mathbb{Z}_p}(G_{\infty})$  is fully faithful.*  
 (2) *A semi-stable representation with Hodge-Tate weights in  $\{0, \dots, r\}$  is of finite  $E(u)$ -height  $r$ .*

**Remark 2.1.2.** (1) Suppose that  $V$  is of  $E(u)$ -height  $r$ . Then it is easy to show that any  $G_{\infty}$ -stable  $\mathbb{Z}_p$ -lattice  $T \subset V$  comes from a Kisin module  $\mathfrak{N} \in \text{Mod}_{\mathfrak{S}}^{r, \text{fr}}$ , i.e.,  $T \simeq T_{\mathfrak{S}}(\mathfrak{N})$ . See the proof of Lemma (2.1.15) in [Kis06].  
 (2) It is natural to ask if the converse question for Theorem 2.1.1 (2) is true. As we will see later, our results in this note may be regarded as partial results in this direction.

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<sup>3</sup>A  $p$ -adic representation  $V$  is called semi-stable if  $\dim_{K_0}(V \otimes_{\mathbb{Q}_p} B_{\text{st}})^G = \dim_{\mathbb{Q}_p} V$ . See [Fon94a] for the construction of  $B_{\text{st}}$ .

**2.2.  $(\varphi, \hat{G})$ -modules.** We denote by  $S$  the  $p$ -adic completion of the divided power envelope of  $W(k)[u]$  with respect to the ideal generated by  $E(u)$ . There is a unique map (Frobenius)  $\varphi : S \rightarrow S$  which extends the Frobenius on  $\mathfrak{S}$ . Define a continuous  $K_0$ -linear derivation  $N : S \rightarrow S$  such that  $N(u) = -u$ . We denote  $S[1/p]$  by  $S_{K_0}$ .

Recall  $R = \varprojlim \mathcal{O}_{\bar{K}}/p$  and the unique surjective map  $\theta : W(R) \rightarrow \widehat{\mathcal{O}_{\bar{K}}}$  which lifts the projection  $R \rightarrow \mathcal{O}_{\bar{K}}/p$  onto the first factor in the inverse limit. We denote by  $A_{\text{cris}}$  the  $p$ -adic completion of the divided power envelope of  $W(R)$  with respect to  $\text{Ker}(\theta)$ . Recall that  $[\pi] \in W(R)$  is the Teichmüller representative of  $\pi = (\pi_n)_{n \geq 0} \in R$  and we embed the  $W(k)$ -algebra  $W(k)[u]$  into  $W(R)$  via  $u \mapsto [\pi]$ . Since  $\theta([\pi]) = \pi$ , this embedding extends to an embedding  $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\text{cris}}$ , and  $\theta|_S$  is the  $K_0$ -linear map  $s : S \rightarrow \mathcal{O}_K$  defined by sending  $u$  to  $\pi$ . The embedding is compatible with Frobenius endomorphisms. As usual, we write  $B_{\text{cris}}^+ = A_{\text{cris}}[1/p]$ .

For any field extension  $F/\mathbb{Q}_p$ , set  $F_{p^\infty} = \bigcup_{n=1}^{\infty} F(\zeta_{p^n})$  with  $\zeta_{p^n}$  a primitive  $p^n$ -th root of unity. Note that  $K_{\infty, p^\infty} = \bigcup_{n=1}^{\infty} K(\pi_n, \zeta_{p^n})$  is Galois over  $K$ . Let  $G_0 := \text{Gal}(K_{\infty, p^\infty}, K_{p^\infty})$ ,  $H_K := \text{Gal}(K_{\infty, p^\infty}, K_\infty)$  and  $\hat{G} := \text{Gal}(K_{\infty, p^\infty}/K)$ . By Lemma 5.1.2 in [Liu07a], we have  $K_{p^\infty} \cap K_\infty = K$ ,  $\hat{G} = G_0 \rtimes H_K$  and  $G_0 \simeq \mathbb{Z}_p(1)$ .

For any  $g \in G$ , let  $\underline{\epsilon}(g) = g([\pi])/[\pi]$ . Then  $\underline{\epsilon}(g)$  is a cocycle from  $G$  to the group of units of  $A_{\text{cris}}$ . In particular, fixing a topological generator  $\tau$  of  $G_0$ , the fact that  $\hat{G} = G_0 \rtimes H_K$  implies that  $\underline{\epsilon}(\tau) = [(\epsilon_i)_{i \geq 0}] \in W(R)$  with  $\epsilon_i$  a *primitive*  $p^i$ -th root of unity. Therefore,  $t := -\log(\underline{\epsilon}(\tau)) \in A_{\text{cris}}$  is well defined and for any  $g \in G$ ,  $g(t) = \chi(g)t$  where  $\chi$  is the cyclotomic character.

For any integer  $n \geq 0$ , let  $t^{\{n\}} = t^{r(n)} \gamma_{\tilde{q}(n)}(t^{p-1}/p)$  where  $n = (p-1)\tilde{q}(n) + r(n)$  with  $0 \leq r(n) < p-1$  and  $\gamma_i(x) = \frac{x^i}{i!}$  is the standard divided power. Define a subring  $\mathcal{R}_{K_0}$  of  $B_{\text{cris}}^+$  as in §6, [Liu07b]:

$$\mathcal{R}_{K_0} = \{x = \sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S_{K_0} \text{ and } f_i \rightarrow 0 \text{ as } i \rightarrow +\infty\}.$$

Finally we put  $\hat{\mathcal{R}} := \mathcal{R}_{K_0} \cap W(R)$ .

It is easy to see that  $\mathcal{R}_{K_0}$  is an  $S$ -algebra and  $\varphi$ -stable as a subring of  $B_{\text{cris}}^+$ . We claim that  $\mathcal{R}_{K_0}$  is also  $G$ -stable. In fact, it suffices to show that for any  $g \in G$ ,  $x \in S$ , we have  $g(x) \in \mathcal{R}_{K_0}$ . First note that for any  $g \in G_\infty$ ,  $g(x) = x$ . Recall that  $\hat{G} = G_0 \rtimes H_K$  and  $\tau$  the fixed topological generator in  $G_0$ . It suffices to check that  $\tau(x) \in \mathcal{R}_{K_0}$ . But we have

$$\tau(x) := \sum_{i=0}^{\infty} N^i(x) \gamma_i(-\log(\underline{\epsilon}(\tau))) = \sum_{i=0}^{\infty} N^i(x) \gamma_i(t).$$

Therefore  $\tau(x) \in \mathcal{R}_{K_0}$  and  $\mathcal{R}_{K_0}$  is  $G$ -stable. In fact, the  $G$  action on  $\mathcal{R}_{K_0}$  factors through  $\hat{G}$ . Hence we obtain some elementary facts on  $\hat{\mathcal{R}}$ .

**Lemma 2.2.1.** (1)  $\hat{\mathcal{R}}$  is a  $\varphi$ -stable  $\mathfrak{S}$ -algebra as a subring in  $W(R)$ .  
 (2)  $\hat{\mathcal{R}}$  is  $G$ -stable. The  $G$ -action on  $\hat{\mathcal{R}}$  factors through  $\hat{G}$ .

*Remark 2.2.2.* By Lemma 7.1.2 in [Liu07b], we may regard  $R_{K_0}$  as a subring of  $K_0[[x, y]]$  via  $u \mapsto x$  and  $t \mapsto y$ . However, the structure of  $\hat{\mathcal{R}}$  is much more complicated and so far we do not know how to describe it explicitly. See Example 3.2.3.

Let  $(\mathfrak{M}, \varphi_{\mathfrak{M}})$  be a Kisin module of height  $r$  and  $\hat{\mathfrak{M}} := \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ . Then we can naturally extend  $\varphi$  from  $\mathfrak{M}$  to  $\hat{\mathfrak{M}}$  by

$$\varphi_{\hat{\mathfrak{M}}}(a \otimes m) = \varphi_{\hat{\mathcal{R}}}(a) \otimes \varphi_{\mathfrak{M}}(m), \quad \forall a \in \hat{\mathcal{R}}, \forall m \in \mathfrak{M}.$$

**Definition 2.2.3.** A  $(\varphi, \hat{G})$ -module (of height  $r$ ) is a triple  $(\mathfrak{M}, \varphi, \hat{G})$  where

- (1)  $(\mathfrak{M}, \varphi_{\mathfrak{M}})$  is a Kisin module (of height  $r$ ).
- (2)  $\hat{G}$  is a  $\hat{\mathcal{R}}$ -semi-linear  $\hat{G}$ -action on  $\hat{\mathfrak{M}} := \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ .
- (3)  $\hat{G}$  commutes with  $\varphi_{\hat{\mathfrak{M}}}$  on  $\hat{\mathfrak{M}}$ , i.e., for any  $g \in \hat{G}$ ,  $g\varphi_{\hat{\mathfrak{M}}} = \varphi_{\hat{\mathfrak{M}}}g$ .
- (4) Regard  $\mathfrak{M}$  as an  $\varphi(\mathfrak{S})$ -submodule in  $\hat{\mathfrak{M}}$ , then  $\mathfrak{M} \subset \hat{\mathfrak{M}}^{H_K}$ .

A morphism between two  $(\varphi, \hat{G})$ -modules is a morphism of Kisin modules and commutes with  $\hat{G}$ -action on  $\hat{\mathfrak{M}}$ 's. We denote by  $\text{Mod}_{\varphi, \mathfrak{S}}^{r, \hat{G}}$  the category of  $(\varphi, \hat{G})$ -modules of height  $r$ .

**2.3. The main theorem.** Let  $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$  be a  $(\varphi, \hat{G})$ -module. We can associate a  $\mathbb{Z}_p[G]$ -module:

$$(2.3.1) \quad \hat{T}(\hat{\mathfrak{M}}) := \text{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, W(R)),$$

where  $G$ -acts on  $\hat{T}(\hat{\mathfrak{M}})$  via  $g(f)(x) = g(f(g^{-1}(x)))$  for any  $g \in G$  and  $f \in \hat{T}(\hat{\mathfrak{M}})$ . Now we can state our main theorem:

**Theorem 2.3.1.** (1) Let  $\hat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G})$  be a  $(\varphi, \hat{G})$ -module. There is a natural isomorphism of  $\mathbb{Z}_p[G_{\infty}]$ -modules

$$(2.3.2) \quad \theta : T_{\mathfrak{S}}(\mathfrak{M}) = \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}^{\text{ur}}) \xrightarrow{\sim} \hat{T}(\hat{\mathfrak{M}}) = \text{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, W(R)).$$

- (2)  $\hat{T}$  induces an anti-equivalence between the category of  $(\varphi, \hat{G})$ -modules of height  $r$  and the category of  $G$ -stable  $\mathbb{Z}_p$ -lattices in semi-stable representations with Hodge-Tate weights in  $\{0, \dots, r\}$ .

### 3. THE PROOF OF THE MAIN THEOREM

**3.1. The connection to Kisin's theory.** We first prove Theorem 2.3.1 (1) and full faithfulness of  $\hat{T}$  in this subsection.

Let  $(\mathfrak{M}, \varphi, \hat{G})$  be a  $(\varphi, \hat{G})$ -module and  $\hat{\mathfrak{M}} := \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ . As in Definition 2.2.3, we regard  $\mathfrak{M}$  as a  $\varphi(\mathfrak{S})$ -submodule of  $\hat{\mathfrak{M}}$ . Then for any  $f \in T_{\mathfrak{S}}(\mathfrak{M})$ , define  $\theta(f) \in \text{Hom}_{\hat{\mathcal{R}}}(\hat{\mathfrak{M}}, W(R))$  by

$$\theta(f)(a \otimes x) := a\varphi(f(x)), \quad \forall a \in \hat{\mathcal{R}}, \forall x \in \mathfrak{M}.$$

It is routine to check that  $\theta(f)$  is well-defined and preserves Frobenius. Therefore,  $\theta : T_{\mathfrak{S}}(\mathfrak{M}) \rightarrow \hat{T}(\hat{\mathfrak{M}})$  is a well-defined. Now we reduce the proof of Theorem 2.3.1 (1) to the following

**Lemma 3.1.1.**  $\theta : T_{\mathfrak{S}}(\mathfrak{M}) \rightarrow \hat{T}(\hat{\mathfrak{M}})$  is an isomorphism of  $\mathbb{Z}_p[G_{\infty}]$ -modules.

*Proof.* Since  $\varphi : \mathfrak{S}^{\text{ur}} \rightarrow W(R)$  is injective,  $\theta$  is obviously an injection. To see that  $\theta$  is surjective, for any  $h \in \hat{T}(\hat{\mathfrak{M}})$ , consider  $f := h|_{\mathfrak{M}}$ . Since  $f$  is a  $\varphi(\mathfrak{S})$ -linear morphism from  $\mathfrak{M}$  to  $W(R) = \varphi(W(R))$ . There exists an  $\mathfrak{f} \in \text{Hom}_{\mathfrak{S}}(\mathfrak{M}, W(R))$  such that  $\varphi(\mathfrak{f}) = f$ . Obviously,  $\theta(\mathfrak{f}) = h$  and  $\mathfrak{f}$  preserves Frobenius. Now we have  $\mathfrak{f} \in \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, W(R))$ . It suffices to show that  $\mathfrak{f} \in T_{\mathfrak{S}}(\mathfrak{M}) = \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}^{\text{ur}})$ . Note that  $f(\mathfrak{M}) \subset W(R)$  is an  $\mathfrak{S}$ -finite type  $\varphi$ -stable submodule and of  $E(u)$ -height  $r$ . By [Fon90], Proposition B 1.8.3, we have  $f(\mathfrak{M}) \subset \mathfrak{S}^{\text{ur}}$ . This complete the proof of the bijection of  $\theta$ . Now it suffices to check that  $\theta$  is compatible with  $G_{\infty}$ -actions on the both sides. For any  $g \in G_{\infty}$ ,  $a \in \hat{\mathcal{R}}$ ,  $x \in \mathfrak{M}$  and  $f \in T_{\mathfrak{S}}(\mathfrak{M})$ ,  $g(\theta(f))(a \otimes x) = g(\theta(f)(g^{-1}(a \otimes x)))$ . Note that  $G_{\infty}$  acts on  $\mathfrak{M}$  trivially, we have

$$g(\theta(f)(g^{-1}(a \otimes x))) = g(\theta(f)(g^{-1}(a) \otimes x)) = a \otimes g(\varphi(f(x))) = \theta(g(f))(a \otimes x).$$

That is,  $g(\theta(f)) = \theta(g(f))$ .  $\square$

Now we need some preparations to show that  $\hat{T}(\hat{\mathfrak{M}})$  is semi-stable. Let  $T$  be a finite free  $\mathbb{Z}_p$ -representation of  $G$  or  $G_{\infty}$ , we denote by  $T^{\vee}$  the  $\mathbb{Z}_p$ -dual of  $T$ . It will be useful to recall the following technical results from [Liu07b], §3.2: let  $\mathfrak{M}$  be a Kisin module of height  $r$ , using the definition of  $T_{\mathfrak{S}}(\mathfrak{M})$ , we can show (c.f. [Liu07b], Proposition 3.2.1) there exists an  $\mathfrak{S}^{\text{ur}}$ -linear,  $G_{\infty}$ -compatible morphism<sup>4</sup>

$$\iota_{\mathfrak{S}} : \mathfrak{S}^{\text{ur}} \otimes_{\mathfrak{S}} \mathfrak{M} \rightarrow T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\text{ur}}.$$

Select a  $\mathfrak{t} \in \mathfrak{S}^{\text{ur}}$  such that  $\varphi(\mathfrak{t}) = c_0^{-1}E(u)\mathfrak{t}$  where  $c_0$  is the constant term of  $E(u)$ . Such  $\mathfrak{t}$  is unique up to units of  $\mathbb{Z}_p$ , see Example 2.3.5 in [Liu07b] for details.

**Lemma 3.1.2.**  *$\iota_{\mathfrak{S}}$  is an injection. If we regard  $\mathfrak{S}^{\text{ur}} \otimes_{\mathfrak{S}} \mathfrak{M}$  as a submodule of  $T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\text{ur}}$  via  $\iota_{\mathfrak{S}}$ . Then  $\mathfrak{t}^r(T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\text{ur}}) \subset \mathfrak{S}^{\text{ur}} \otimes_{\mathfrak{S}} \mathfrak{M}$ .*

*Proof.* See Theorem 3.2.2 in [Liu07b].  $\square$

Using the same idea as above, we have a similar result for  $\hat{\mathfrak{M}}$

**Proposition 3.1.3.**  *$\hat{T}(\hat{\mathfrak{M}})$  induces a natural  $W(R)$ -linear,  $G$ -compatible morphism*

$$(3.1.1) \quad \hat{\iota} : W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} \longrightarrow \hat{T}^{\vee}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R),$$

where  $\hat{\mathfrak{M}} = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ . Moreover,  $\hat{\iota} = \iota_{\mathfrak{S}} \otimes_{\mathfrak{S}^{\text{ur}}, \varphi} W(R)$  and  $\hat{\iota}$  is an injection. If we regard  $W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}$  as a submodule of  $\hat{T}^{\vee}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R)$  via  $\hat{\iota}$ . Then  $(\varphi(\mathfrak{t}))^r(\hat{T}^{\vee}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R)) \subset W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}$ .

*Proof.* We use the same idea for the construction of  $\iota_{\mathfrak{S}}$  in Proposition 3.2.1 in [Liu07b]. One first prove that

$$\hat{T}(\hat{\mathfrak{M}}) \simeq \text{Hom}_{W(R), \varphi}(W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}, W(R))$$

is an isomorphism of  $G$ -modules, where the  $G$ -action on the right side is given by  $g(f)(\cdot) = g(f(g^{-1}(\cdot)))$ , for any  $g \in G$  and  $f \in \text{Hom}_{W(R), \varphi}(W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}, W(R))$ . Then we have a map

$$\hat{\iota} : W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} \rightarrow \text{Hom}_{\mathbb{Z}_p}(\hat{T}(\hat{\mathfrak{M}}), W(R)) = \hat{T}^{\vee}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R)$$

induced by  $x \mapsto (f \mapsto f(x), \forall f \in \hat{T}(\hat{\mathfrak{M}}))$  for any  $x \in \hat{\mathfrak{M}}$ . It is easy to check that  $\hat{\iota}$  is compatible with  $G$ -actions on the both sides. By Lemma 3.1.1 and comparing

<sup>4</sup>Here we use a slightly different notations from those in [Liu07b].

the constructions of  $\iota_{\mathfrak{S}}$  and  $\hat{\iota}$ , we see that  $\hat{\iota} = \iota_{\mathfrak{S}} \otimes_{\mathfrak{S}^{\text{ur}}, \varphi} W(R)$ . The remaining statements are then easy consequences of Lemma 3.1.2.  $\square$

*Remark 3.1.4.* Let  $V$  be a representation of  $E(u)$ -height  $r$ ,  $T$  a  $G$ -stable  $\mathbb{Z}_p$ -lattice in  $V$ , and  $\mathfrak{M}$  the Kisin module associated to  $T|_{G_{\infty}}$ . We can always consider the injection

$$\tilde{\iota} := \iota_{\mathfrak{S}} \otimes_{\mathfrak{S}^{\text{ur}}, \varphi} W(R) : W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \hookrightarrow T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} W(R).$$

There is a natural  $G$ -action on the right side because  $T$  is  $G$ -stable. In general, it is not clear whether the left side is  $G$ -stable<sup>5</sup>, or equivalently, whether the  $G$ -orbit of  $\mathfrak{M}$ ,  $G(\mathfrak{M}) \subset W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ . As we will see soon, in the case of  $(\varphi, \hat{G})$ -modules, we have  $G(\mathfrak{M}) \subset \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \subset W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ . This is actually a key point to prove that  $\hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  is semi-stable.

Now we are ready to prove that  $\hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  is semi-stable. Tensoring  $B_{\text{cris}}^+$  on both sides of (3.1.1), noting that

$$B_{\text{cris}}^+ \otimes_{W(R)} W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} = B_{\text{cris}}^+ \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} = B_{\text{cris}}^+ \otimes_{\mathcal{R}_{K_0}} \mathcal{R}_{K_0} \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}$$

and  $\mathcal{R}_{K_0} \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} \simeq \mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ , we have

$$(3.1.2) \quad \hat{\iota} \otimes_{W(R)} B_{\text{cris}}^+ : B_{\text{cris}}^+ \otimes_{\mathcal{R}_{K_0}} (\mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) \rightarrow \hat{T}^{\vee}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} B_{\text{cris}}^+.$$

By a similar argument for  $\iota_{\mathfrak{S}}$ , we also have

$$(3.1.3) \quad \iota_{\mathfrak{S}} \otimes_{\mathfrak{S}^{\text{ur}}, \varphi} B_{\text{cris}}^+ : B_{\text{cris}}^+ \otimes_{\mathcal{R}_{K_0}} (\mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) \rightarrow T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} B_{\text{cris}}^+.$$

Since  $\hat{\iota} = \iota_{\mathfrak{S}} \otimes_{\mathfrak{S}^{\text{ur}}, \varphi} W(R)$  by Proposition 3.1.3, we have the following commutative diagram to identify (3.1.2) with (3.1.3):

$$\begin{array}{ccc} B_{\text{cris}}^+ \otimes_{\mathcal{R}_{K_0}} (\mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) & \xrightarrow{(3.1.2)} & \hat{T}^{\vee}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} B_{\text{cris}}^+ \\ \downarrow \wr & & \downarrow \wr \\ B_{\text{cris}}^+ \otimes_{\mathcal{R}_{K_0}} (\mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) & \xrightarrow{(3.1.3)} & T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} B_{\text{cris}}^+ \end{array}$$

Thus  $\mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$  in (3.1.3) is  $G$ -stable and has the same  $\hat{G}$ -action as that on  $\mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$  in (3.1.2). Now the proof of semi-stability of  $\hat{T}(\hat{\mathfrak{M}})$  will be totally same as [Liu07b], §7. For convenience of readers, we sketch the proof here.

[Liu07b], §7 is also aiming to prove that certain representation  $V$  of  $E(u)$ -height  $r$  is semi-stable with Hodge-Tate weights in  $\{0, \dots, r\}$ . Except that we require that  $V$  is of finite  $E(u)$ -height such that we can establish (3.1.3), the only other inputs that §7 need are three conditions in the beginning of §7.1 on the  $\hat{G}$ -action on  $\mathcal{D} \otimes_S \mathcal{R}_{K_0} \simeq \mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ , where  $\mathcal{D} := S_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ . And these three conditions are just conditions required in Definition 2.2.3. Thus the same proof follows. More precisely, regarding  $S_{K_0}$  as a subring of  $K_0[[u]]$ , let  $I = uK_0[[u]] \cap S_{K_0}$  and  $D := \mathcal{D}/I\mathcal{D}$ . Then  $D$  is a finite free  $K_0$ -module with a semi-linear Frobenius action. One can prove there is a unique  $\varphi$ -equivariant section  $D \hookrightarrow \mathcal{D}$ . So we can regard  $D$  as a  $K_0$ -submodule in  $\mathcal{D}$ . Since  $D \hookrightarrow \mathcal{D}$  is  $\varphi$ -equivariant, the structure of  $\mathcal{R}_{K_0}$  forces that  $\hat{G}(D) \subset K_0[t] \otimes_{K_0} D$ . Now the fact that  $\hat{G}$  acts on  $K_0[t] \otimes_{K_0} D$  and  $H_K$

<sup>5</sup>Though it is  $G_{\infty}$ -stable.

acts on  $D$  trivially implies that there exists a linear map  $N : D \rightarrow D$  such that  $\tau(x) = \sum_{n=0}^{\infty} \gamma_i(t) \otimes N^i(x)$  for any  $x \in D$ . Now consider the  $K_0$ -vector space

$$\bar{D} := \left\{ \sum_{i=0}^{\infty} \gamma_i(u) \otimes N^i(x) \in B_{\text{st}}^+ \otimes_S \mathcal{D} \mid x \in D \right\}$$

where  $u = \log(u) \in B_{\text{st}}^+$ . We can show that  $\bar{D} \subset (B_{\text{st}}^+ \otimes_S \mathcal{D})^G \subset (\hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} B_{\text{st}}^+)^G$ . But  $\dim_{K_0} \bar{D} = \dim_{K_0} D = \text{Rank}_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})$ . Therefore,  $\hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  is semi-stable and the functor  $\hat{T}$  is well-defined.

Now let us prove the full faithfulness of  $\hat{T}$ . Suppose that  $f : T' \rightarrow T$  is a morphism of  $G$ -stable  $\mathbb{Z}_p$ -lattices inside semi-stable representations, and there exist  $(\varphi, \hat{G})$ -modules  $\hat{\mathfrak{M}}'$  and  $\hat{\mathfrak{M}}$  such that  $\hat{T}(\hat{\mathfrak{M}}') \simeq T'$  and  $\hat{T}(\hat{\mathfrak{M}}) \simeq T$ . Note that  $T_{\mathfrak{S}}$  is fully faithful (Theorem 2.1.1), there exists a morphism of Kisin modules  $\mathfrak{f} : \mathfrak{M} \rightarrow \mathfrak{M}'$  such that  $T_{\mathfrak{S}}(\mathfrak{f}) = f|_{G_\infty}$ , where  $\hat{\mathfrak{M}} = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$  and  $\hat{\mathfrak{M}}' = \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}'$ . By Lemma 3.1.1, it suffices to show that  $\hat{\mathfrak{f}} := \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{f}$  is  $G$ -equivariant. To see this, consider the following commutative diagram induced by  $\hat{\mathfrak{f}}$  defined in (3.1.1):

$$\begin{array}{ccccc} W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} & \xrightarrow{\hat{\mathfrak{f}}} & T^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R) & \xlongequal{\quad} & T^\vee \otimes_{\mathbb{Z}_p} W(R) \\ \downarrow W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{f}} & & \downarrow T_{\mathfrak{S}}(\mathfrak{f}) \otimes_{\mathbb{Z}_p} W(R) & & \downarrow f \otimes_{\mathbb{Z}_p} W(R) \\ W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}' & \xrightarrow{\hat{\mathfrak{f}}'} & T^\vee(\hat{\mathfrak{M}}') \otimes_{\mathbb{Z}_p} W(R) & \xlongequal{\quad} & (T')^\vee \otimes_{\mathbb{Z}_p} W(R) \end{array}$$

Note that  $\hat{\mathfrak{f}}$  is injective by Proposition 3.1.3. Since  $f$  is  $G$ -equivariant,  $\hat{\mathfrak{f}}$  is  $G$ -equivariant.

**3.2. The essential surjectiveness of  $\hat{T}$ .** Now assume  $T$  is a  $G$ -stable  $\mathbb{Z}_p$ -lattice in a semi-stable representation  $V$  with Hodge-Tate weights in  $\{0, \dots, r\}$ . By Theorem 2.1.1. There exists a Kisin module  $\mathfrak{M}$  such that  $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T|_{G_\infty}$ . Theorem 5.4.2 in [Liu07b] showed that

$$\iota_{\mathfrak{S}} \otimes_{\mathfrak{S}^{\text{ur}}, \varphi} B_{\text{cris}}^+ : B_{\text{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \longrightarrow T_{\mathfrak{S}}^\vee(\mathfrak{M}) \otimes_{\mathbb{Z}_p} B_{\text{cris}}^+ = T^\vee \otimes_{\mathbb{Z}_p} B_{\text{cris}}^+$$

is compatible with  $G$ -action. More precisely, let  $\mathcal{D} := S_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$  be the Breuil module<sup>6</sup> associated to  $V$  and  $N$  be the monodromy operator on  $\mathcal{D}$ . Then  $G$  acts on  $B_{\text{cris}}^+ \otimes_S \mathcal{D} = B_{\text{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$  via (c.f. (5.2.1) in [Liu07b])

$$(3.2.1) \quad g(a \otimes x) = \sum_{i=0}^{\infty} g(a) \gamma_i(-\log(\underline{\epsilon}(g))) \otimes N^i(x).$$

We identify  $\mathfrak{M}$  as a  $\varphi(\mathfrak{S})$ -submodule of  $B_{\text{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$  by

$$\mathfrak{M} \simeq \mathfrak{S} \otimes_{\mathfrak{S}} \mathfrak{M} \hookrightarrow \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \hookrightarrow B_{\text{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}.$$

Now we are interested in the orbit  $G(\mathfrak{M})$  of  $\mathfrak{M}$  under  $G$ .

**Proposition 3.2.1.**  $G(\mathfrak{M}) \subset \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ .

<sup>6</sup>A Breuil module is a finite free  $S_{K_0}$ -module with structures of Frobenius, filtration and monodromy. By [Bre97], the category of admissible Breuil modules is equivalent to the category of semi-stable representations. Also see §3.2 in [Liu07a] for the relation between Kisin modules and Breuil modules.



*Proof.* We have seen that  $G(\mathfrak{M}) \subset \mathcal{R}_{K_0} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$  by formula (3.2.1). Then it suffices to show that  $\tau(\mathfrak{M}) \subset W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ , where  $\tau$  is the fixed generator of  $G_0$ . Now consider the following commutative diagram

$$(3.2.2) \quad \begin{array}{ccccc} \mathfrak{M} & \xrightarrow{\varphi \otimes 1} & W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} & \longrightarrow & T^\vee \otimes_{\mathbb{Z}_p} W(R) \\ \parallel & & \downarrow & & \downarrow \\ \mathfrak{M} & \xrightarrow{\varphi \otimes 1} & B_{\text{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} & \longrightarrow & T^\vee \otimes_{\mathbb{Z}_p} B_{\text{cris}}^+ \end{array}$$

where the first row is obtained by  $\iota_{\mathfrak{S}} \otimes_{\mathfrak{S}^{\text{ur}}, \varphi} W(R)$ . Obviously, the right column is compatible with  $G$ -action. By Proposition 3.1.3, we have

$$(\varphi(\mathfrak{t}))^r(\tau(\mathfrak{M})) \subset W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}.$$

Now select a basis  $e_1, \dots, e_d$  of  $\mathfrak{M}$  and write  $\tau(e_1, \dots, e_d) = (e_1, \dots, e_d)A$  with  $A$  a  $d \times d$ -matrix. Let  $a$  be a coefficient of  $A$ . It suffices to show that  $a \in W(R)$ . Now we know that  $a \in \mathcal{R}_{K_0}$  and  $(\varphi(\mathfrak{t}))^r a \in W(R)$ . Then we may reduce the proof to Lemma 3.2.2 below.  $\square$

**Lemma 3.2.2.** *Let  $a \in B_{\text{cris}}^+$ . If  $(\varphi(\mathfrak{t}))^r a \in W(R)$  then  $a \in W(R)$ .*

*Proof.* After multiplying some  $p$ -power, we may assume that  $a \in A_{\text{cris}}$ . As in [Fon94a], §5.1, define

$$I^{[r]}W(R) = \{a \in W(R) \mid \varphi^n(a) \in \text{Fil}^r W(R), \text{ for any } n \geq 0\}.$$

Write  $x = (\varphi(\mathfrak{t}))^r a$ . We claim that  $x \in I^{[r]}W(R)$ . By Example 5.3.3 in [Liu07b] or Example 3.2.3 below, there exists a unit  $\alpha \in A_{\text{cris}}$  such that  $t = \alpha\varphi(\mathfrak{t})$ . So  $\varphi(\mathfrak{t}) \in \text{Fil}^1 W(R)$ , then  $x = (\varphi(\mathfrak{t}))^r a \in \text{Fil}^r A_{\text{cris}} \cap W(R) = \text{Fil}^r W(R)$ . On the other hand,  $\varphi(\mathfrak{t}) = c_0^{-1}E(u)\mathfrak{t}$ , thus  $\varphi^n(\mathfrak{t}) = (\prod_{i=0}^{n-1} \varphi^i(c_0^{-1}E(u)))\mathfrak{t} \in \text{Fil}^1 W(R)$ . Therefore  $\varphi^n(x), \varphi^n((\varphi(\mathfrak{t}))^r) \in \text{Fil}^r W(R)$  and then  $x, (\varphi(\mathfrak{t}))^r \in I^{[r]}W(R)$ . By proposition 5.1.3 in [Fon94a],  $I^{[r]}W(R)$  is a principal ideal and  $b \in I^{[r]}W(R)$  is a generator if and only if  $v_R(\tilde{b}) = \frac{rp}{p-1}$ , where  $\tilde{b} = b \bmod p$ . We claim that  $(\varphi(\mathfrak{t}))^r$  is a generator of  $I^{[r]}W(R)$  by computing  $v_R(\widetilde{(\varphi(\mathfrak{t}))^r}) = \frac{p}{p-1}$ . Since  $\varphi(\mathfrak{t}) = c_0^{-1}E(u)\mathfrak{t}$ , we may choose  $\mathfrak{t}$  such that  $\tilde{\mathfrak{t}} = \mathfrak{t} \bmod p = \frac{1}{p-1}$ . Thus  $v_R(\tilde{\mathfrak{t}}) = \frac{1}{p-1}$ . Now  $(\varphi(\mathfrak{t}))^r a \in I^{[r]}W(R)$  and  $(\varphi(\mathfrak{t}))^r$  is a generator of  $I^{[r]}W(R)$ . So  $a \in W(R)$ .  $\square$

Now Proposition 3.2.1 implies that  $\hat{\mathfrak{M}} := \hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$  is stable under  $G$ -action in  $B_{\text{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \hookrightarrow T^\vee \otimes_{\mathbb{Z}_p} B_{\text{cris}}^+$ . And obviously the  $G$ -action on  $\hat{\mathfrak{M}}$  factors through  $\hat{G}$ . It is easy to check that  $\hat{\mathfrak{M}}$  is a  $(\varphi, \hat{G})$ -module. It remains to check that  $\hat{T}(\hat{\mathfrak{M}}) \simeq T$ . First, by Lemma 3.1.1,  $\hat{T}(\hat{\mathfrak{M}})|_{G_\infty} \simeq T|_{G_\infty}$ . Recall that  $\hat{\iota}$  defined in (3.1.1) is compatible with the  $G$ -action on the both sides, and  $\hat{\iota} = \iota_{\mathfrak{S}} \otimes_{\mathfrak{S}, \varphi} W(R)$ . Comparing  $\hat{\iota}$  with the top row of (3.2.2), we have the following commutative diagram:

$$\begin{array}{ccc} W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}} & \xrightarrow{\hat{\iota}} & \hat{T}^\vee(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} W(R) \\ \downarrow \wr & & \downarrow \wr \\ W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} & \longrightarrow & T^\vee \otimes_{\mathbb{Z}_p} W(R) \end{array}$$

By the construction of  $\hat{\mathfrak{M}}$ , we see that the left column is compatible with the  $G$ -actions. By Proposition 3.1.3,  $\varphi(t^r)(T_{\mathfrak{S}}^{\vee}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} W(R)) \subset W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ . So the right column is also compatible with the  $G$ -actions. Therefore,  $\hat{T}(\mathfrak{M}) \simeq T$  as  $G$ -modules. This completes the proof the main theorem.

Unlike  $\mathcal{R}_{K_0}$ , so far we do not have an explicit description of  $\hat{\mathcal{R}}$ . Put

$$w := \frac{\tau(u)}{u} - 1 = \frac{\tau([\pi])}{[\pi]} - 1 = \exp(-t) - 1.$$

We see that  $w \in \hat{\mathcal{R}}$  and  $W(k)[[u, w]] \subset \hat{\mathcal{R}}$  is stable under Frobenius and  $\hat{G}$ -action. Unfortunately, this inclusion is strict. The following example show that the structure of  $\hat{\mathcal{R}}$  may be very complicated.

**Example 3.2.3.** It is well known that  $t$  is the period of the cyclotomic character  $\chi$ . On the other hand,  $\mathfrak{t}$  is the period of the Kisin module for  $\chi$ , where is  $\mathfrak{S}$ -free rank-1 module  $\mathfrak{S}^* := \mathfrak{S} \cdot f$  and  $\varphi(f) = c_0^{-1}E(u)f$  with  $f$  a basis. Example 5.3.3 in [Liu07b] showed that we may choose  $t$  such that  $t = c\varphi(\mathfrak{t})$ , where  $c = \prod_{i=0}^{\infty} \varphi^n(\frac{\varphi(c_0^{-1}E(u))}{p})$ . Then  $\tau(c)\tau(\varphi(\mathfrak{t})) = \tau(t) = t = c\varphi(\mathfrak{t})$ . Therefore  $\tau(\varphi(\mathfrak{t})) = \frac{c}{\tau(c)}\varphi(\mathfrak{t}) = \prod_{n=1}^{\infty} \varphi^n(\frac{E(u)}{\tau(E(u))})\varphi(\mathfrak{t})$ . Let  $\hat{c} = \frac{c}{\tau(c)} = \prod_{n=1}^{\infty} \varphi^n(\frac{E(u)}{\tau(E(u))})$ . Since  $c$  is a unit in  $\mathcal{R}_{K_0}$ ,  $\hat{c} \in \mathcal{R}_{K_0}$ . On the other hand,  $E(u)$  is a generator of  $\text{Fil}^1 W(R)$ , so  $\frac{E(u)}{\tau(E(u))}$  is a unit in  $W(R)$ . Thus  $\hat{c} \in \hat{\mathcal{R}}$ . Let  $\hat{\mathfrak{S}}^* := (\hat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}) \cdot f = \hat{\mathcal{R}} \cdot f$  be the  $(\varphi, \hat{G})$ -module corresponding to  $\chi$ . Then  $\hat{G}$ -action on  $\hat{\mathfrak{S}}^*$  is given by  $\tau(f) = \hat{c}f$ .

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